

Non-Abelian Fourier transform for gauge theories

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We consider $SU(2)$ gauge potentials over a space with a compactified dimension. A non-Abelian Fourier transform of the gauge potential in the compactified dimension is defined in such a way that the Fourier coefficients are (almost) gauge invariant. The functional measure and the gauge field strengths are expressed in terms of these Fourier coefficients. The emerging formulation of the non-Abelian gauge theory turns out to be an Abelian gauge theory of a set of fields defined over the initial space with the compactified dimension excluded. The Abelian theory contains an Abelian gauge field, a scalar field, and an infinite tower of vector matter fields, some of which carry Abelian charges. Possible applications of this formalism are discussed briefly.

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I. INTRODUCTION

Despite the years of effort, non-Abelian gauge theories have resisted attempts to obtain a mathematically rigorous, or operationally efficient definition beyond perturbation theory. A particular way to illustrate this assessment is to check what is presently known beyond perturbation theory about the ultraviolet renormalization of gluodynamics (see Refs. [1,2] for recent studies). Simply stated, to date we have only a reasonable expectation that the theory exhibits asymptotic freedom at weak coupling.

This state of affairs justifies further efforts to reformulate the theory and reveal nontrivial observables that may be computable beyond perturbation theory. By nontrivial observables we mean the observables that exhibit nontrivial ultraviolet behavior characteristic of the dynamics.

Candidates for this role can be found in Ref. [3]. In this paper, the light-front formulation of the non-Abelian gauge theories has been reconsidered. A conclusion has been made that there are observables whose dynamics can be split into sectors, each sector representing quantum mechanics of a finite number of degrees of freedom. If this conclusion is true, we indeed have found the desired observables, since many practical methods can be used to solve quantum mechanics with a finite number of degrees of freedom.

However, the conclusion of Ref. [3] is not in agreement with some other results (for a discussion and references, see Ref. [3]). The competing view holds that the light-front formulation is plagued with complications related to the treatment of the zero modes.

The formulation of Ref. [3] is tied to the Hamiltonian framework. While suitable for setting the stage, it is not necessarily the most convenient for implementing the expected symmetries.

In this paper, we avoid reference to the light-front formulation and formulate the non-Abelian Fourier transform of the gauge potentials implicitly present in Ref. [3]. The key feature of the emerging Fourier coefficients of the gauge po-

tentials is that they are almost gauge invariant.

The source of the residual gauge noninvariance of the Fourier coefficients is a compactification of the direction along which we consider the Fourier transform. We compactify along this direction because we want to have a discrete Fourier series. In this case, when a direction of the space-time is compactified, there are gauge transformations that leave the component of the gauge field along the compactified direction intact. Namely, those transformations act nontrivially on the Fourier coefficients we define.

In the next section, we consider a non-Abelian gauge field [to be specific, an $SU(2)$ gauge field A_μ] over a manifold compactified in a selected direction. Using the known facts about the covariant derivative in the compactified direction, and the Wilson loop embracing the compactified direction, we determine a set of almost gauge invariant Fourier components of the gauge potential. The standard formal functional measure $\Pi_{\mu,z} dA_\mu(z)$ is expressed in terms of the Fourier coefficients.

In the third section, we express the field strengths $F_{\mu\nu}$ in terms of the new variables. With these tools, we can express the entire gauge theory in terms of the new variables. The emerging formulation of the non-Abelian gauge theory turns out to be an Abelian gauge theory of fields defined over the initial space-time with the compactified direction excluded.

The field content of the new formulation is as follows. There is an Abelian gauge field, a scalar field (the trace of the Wilson loop embracing the compactified direction), and an infinite tower of vector matter fields some of which are carrying Abelian charges.

Here we mention that resolving an Abelian theory inside a non-Abelian theory is an important ingredient in a confinement scenario [4] (see also Ref. [5] for a recent discussion). We also point out an analogy between our formulation and the so-called Abelian projection formulation [6]. The Abelian gauge symmetry of the Abelian projection formulation is similar to the Abelian gauge symmetry of our formulation.

In the last section, we conclude with a discussion of the

prospects for employing the non-Abelian Fourier transform with the light-front formulation and, in particular, for calculations of the nonperturbative ultraviolet renormalization of gluodynamics. As our formulation is gauge invariant, we are able to formulate a gauge invariant ultraviolet regularization in a manner alternative to Ref. [9].

II. THE FOURIER DATA AND THE MEASURE

Consider a space $S^1 \times M$ and the gauge potential over it, $A_\mu = (A, A_i)$. Let the coordinate along the circle S^1 be x and the manifold M has y^i as its coordinates. Let also the component $A(x, y)$ define the parallel transport along the circle, and the components of the potential $A_i(x, y)$ define the parallel transport along the curves in M .

To be specific, we consider only the case of $SU(2)$ gauge fields. We do not expect any complication in generalizing to $SU(N)$. So, A_μ are traceless, Hermitian, 2×2 matrix functions periodic in x .

Consider the space of matrix-valued functions with the scalar product

$$\langle \Phi | \Psi \rangle = \int dx \, 2 \, \text{Tr}[\Phi^\dagger \Psi]. \quad (1)$$

The dagger signifies Hermitian conjugation and the normalizing factor 2 makes the Pauli matrixes divided by two orthonormal at a unit length of the circle. Our functions will depend on the location y^i on M , but this is considered as a parametric dependence of the functions defined over the circle.

With respect to this scalar product, the operator iD of the covariant derivative along the x direction is Hermitian. Explicitly, $iD\Phi = i\partial_x \Phi + g[A, \Phi]$. The characteristic properties of the spectrum of iD for the circle of length L are as follows.

It is a spectrum of the conventional derivative on a circle with each level split into three levels [the case of $SU(2)$].

One of the levels of each triple stays as if it were a conventional derivative, i.e., it is $2\pi n/L$ with integer n .

Another level of the triple goes above the value $2\pi n/L$ by the increment ϕ/L .

The last level of the triple goes down by the decrement ϕ/L .

The splitting parameter ϕ never goes beyond 2π . It is related to the value of the Wilson loop $w \equiv \text{Tr}[\text{P exp}(ig \oint dx A)]/2$ (P denotes the path ordering) in the following way: $\phi = 2 \arccos(w)$ (we assume that the arccos takes the values in the range from 0 to π). Note that the levels forming the triple above are not always selected from the spectrum by the condition that they are close to one another.¹ When w approaches zero from the positive side, the upper level of the triple associated with the level $2\pi n/L$ collides with the lower level of the triple associated with the

level $2\pi(n+1)/L$. As w further decreases, the upper level of the triple associated with $2\pi n/L$ approaches the value $2\pi(n+1)/L$.

The above facts can be checked using explicit formulas of Ref. [3]. Evidently, the spectrum of iD is gauge invariant (when the potential A undergoes a gauge transformation, the spectrum of iD remains intact). We denote the eigenvalues $p(n, \sigma) = 2\pi n/L + \sigma\phi/L$, or just p for brevity, $\sigma = (-1, 0, +1)$.

An orthonormal set of eigenfunctions $\chi_{p(n, \sigma)}$ can be chosen in such a way that its members transform uniformly under the gauge transformations. This follows from the fact that a covariant derivative of a uniformly transforming object transforms in the same uniform way. Therefore, if χ_p is a normalized eigenfunction of iD at a certain gauge, $U\chi_p U^\dagger$ is a normalized eigenfunction of the covariant derivative involving the gauge transformed field $U[A - \partial_x/(ig)]U^\dagger$.

We use the above set of normalized eigenfunctions χ_p to construct gauge invariants. For example, $F_{\mu\nu}^p \equiv \langle \chi_p | F_{\mu\nu} \rangle$ is a gauge invariant, because the scalar product (1) involves the trace, and both χ_p and the field strength tensor $F_{\mu\nu}$ are transformed uniformly. The above notation with the superscript p for the projection of a field onto the eigenvector χ_p will be in regular use throughout the text.

We first consider the component A of the gauge field along the circle as given, and use it to construct gauge invariants containing all the information about the rest of the gauge field components A_i . We decompose the fields A_i :

$$A_i = A_i^\perp + A_i^\parallel, A_i^\perp \equiv \sum_{p \neq 0} \chi_p A_i^p, A_i^\parallel \equiv \chi_0 A_i^0. \quad (2)$$

In words, we separate the zero mode component (the component along the eigenvector of iD with zero eigenvalue) from the rest of the components. We will treat A_i^\perp and A_i^\parallel differently.

For treating A_i^\perp , we construct from A and χ_p an object living in the subspace transverse to the zero mode and transforming under the gauge transformations exactly in the way the field A_i^\perp does. Indeed, let us check the gauge transformation of

$$B_i \equiv \sum_{p \neq 0} \frac{i}{p} \chi_p \langle \chi_p | \partial_i A \rangle. \quad (3)$$

Consider the infinitesimal gauge transformation $\delta_\alpha A = D\alpha$. It generates the following transformation of the coefficients $\langle \partial_i A \rangle^p$ of the expansion (3):

$$\begin{aligned} \delta_\alpha \langle \chi_p | \partial_i A \rangle &= \langle -ig[\chi_p, \alpha] | \partial_i A \rangle + \langle \chi_p | -ig[\partial_i A, \alpha] \rangle \\ &\quad + \langle \chi_p | D\partial_i \alpha \rangle. \end{aligned} \quad (4)$$

Here, the first term in the right hand side (RHS) is the transformation of χ_p , and the last two terms correspond to the two ways of placing the partial derivative in $\partial_i D\alpha$. Clearly, the first two terms in the RHS cancel against each other, and D in the last term can be replaced by the factor p/i matching the factor i/p in Eq. (3).

¹This is the case though for the positive values of w (here w ranges from -1 to $+1$).

The net result for the gauge transformation of the components B_i^p is as follows:

$$\delta_\alpha B_i^p = \langle \chi_p | \partial_i \alpha \rangle. \quad (5)$$

This is exactly how the components of the gauge field are transformed:

$$\delta_\alpha A_i^p = \langle \chi_p | \partial_i \alpha \rangle. \quad (6)$$

We conclude that

$$\Psi_i^p \equiv A_i^p - B_i^p \quad (7)$$

are gauge invariant. These are the variables we suggest as part of our complete set of new variables. As they are obtained by a shift from the variables A_i^p , there is no Jacobian involved in changing over from A_i^p variables to the Ψ_i^p variables in the functional integral.²

Now let us treat A_i^0 [see Eq. (2)]. Namely, consider the zero modes of A_i , $A_i^0 = \langle \chi_0 | A_i \rangle$. We want to find gauge invariants that would be in one-to-one correspondence with A_i^0 . To do it, let us form a new object out of B_i [see Eq. (3) for the definition] and A_i^0 :

$$C_i \equiv B_i + \chi_0 A_i^0. \quad (8)$$

The point here is that C_i transforms under the gauge transformations exactly as a gauge field

$$\delta_\alpha C_i = D_i[C] \alpha, \quad (9)$$

where $D_i[C] \alpha = \partial_i \alpha - ig[C_i, \alpha]$. It follows from the fact that the nonzero modes of C_i coincide with the nonzero modes of B_i , whose transformation we have studied above. Therefore, for any p (including $p=0$) the transformation of the components of C_i is as it should be for a gauge field $\delta_\alpha C_i^p = \langle \chi_p | \partial_i \alpha \rangle$, which implies Eq. (9).

Using this C_i , we define another gauge invariant

$$G_i \equiv \frac{\sqrt{L} \sigma_p}{g} \langle \chi_p | i D_i[C] \chi_p \rangle. \quad (10)$$

The fact that it is a gauge invariant is implied by the transformation properties of the field C_i mentioned above. The factor in the RHS involving the sign σ_p is chosen to simplify G_i ; σ_p in that factor is determined by p : $p = 2\pi n_p/L + \sigma_p \phi/L$. Naturally, Eq. (10) is valid only for $\sigma_p \neq 0$. As we will see, G_i defined in this way is independent of p as soon as $\sigma_p \neq 0$.

²There is a subtlety here. We imply that χ_p is independent of A_i and is determined by A . It contradicts the requirement that χ_p are transformed uniformly: they do not change if A does not change. Therefore, our new variables are gauge invariant only with respect to the gauge transformations that transform A nontrivially. The left-over gauge transformations of the new variables show up as an Abelian gauge symmetry of the theory formulated in terms of the new variables.

The key property of G_i is that it is uniquely related to the zero mode A_i^0 . To see this, we substitute the definition of C_i into the definition of G_i , and obtain

$$G_i = \frac{\sqrt{L} \sigma_p}{g} (\langle \chi_p | i \partial_i \chi_p \rangle + A_i^0 \langle \chi_p | g[\chi_0, \chi_p] \rangle). \quad (11)$$

The part linear in B_i does not contribute because it is a sum of terms proportional to $\langle \chi_p | [\chi_{p'}, \chi_p] \rangle$ at $p' \neq 0$, which is zero. This is the case because the above commutator, if nonzero, is an eigenfunction of iD with the eigenvalue $p' + p$, which is necessarily orthogonal to χ_p .

We further clarify what is G_i by computing the commutator in the RHS of Eq. (11):

$$[\chi_0, \chi_p] = \frac{\sigma_p}{\sqrt{L}} \chi_p. \quad (12)$$

The easiest way to obtain this expression is to go over to the gauge in which A is diagonal and independent of x . In this gauge,

$$\begin{aligned} \chi_{p(n,0)} &= \frac{\exp\left(-i \frac{2\pi n}{L} x\right) \sigma^3}{2\sqrt{L}}, \\ \chi_{p(n,\pm)} &= \frac{\exp\left(-i \frac{2\pi n}{L} x\right) (\sigma^1 \pm i \sigma^2)}{2\sqrt{2L}}. \end{aligned} \quad (13)$$

It can be explicitly checked that these are indeed the desired eigenfunctions of iD if A is diagonal and independent of x . And Eq. (12) holds in this gauge. It remains to notice that the relation (12) is gauge invariant to conclude that Eq. (12) is valid in any gauge.

Plugging Eq. (12) into Eq. (11) we obtain that

$$G_i = \frac{\sqrt{L} \sigma_p}{g} \langle \chi_p | i \partial_i \chi_p \rangle + A_i^0. \quad (14)$$

The RHS of the above equation formally depends on p , but in fact it does not. To see this, again go over to the gauge where the eigenfunctions χ_p are given by Eq. (13). In this gauge, the first term in the RHS of Eq. (14) disappears, because the eigenfunctions (13) are independent of y^i . Since G_i is gauge invariant, checking that it is independent of p in any gauge suffices to conclude that it is indeed independent of p .

We conclude that G_i is, on the one hand, gauge invariant, and, on the other hand, coincides with A_i^0 up to a shift defined in terms of χ_p . Ultimately, the set of the Fourier data obtained by applying the Fourier transform over the circle to A , A_i are w (the Wilson loop along the compactified direction), Ψ_i^p , and G_i [see Eqs. (7) and (10) for the definitions]. We suggest to use the Fourier data as a new set of the field variables. The new field variables constitute an infinite set of fields over M .

At this point we mention that, in analogy with the ordinary Fourier transform of a real function, the Fourier com-

ponents Ψ_i^p of the Hermitian matrix field $\Psi_i \equiv A_i^\dagger - B_i$ satisfy the condition of complex conjugation $\Psi_i^{-p} = (\Psi_i^p)^\dagger$, where the dagger means complex conjugation.

Next, we derive the measure of the functional integral in terms of the new variables with the following chain of variable transformations:

$$\{A, A_i\} \rightarrow \{A, \Psi_i^p, G_i\} \rightarrow \{w, \Psi_i^p, G_i\}. \quad (15)$$

The first transformation does not introduce any Jacobian, since it is a rotation and a shift of an orthonormal base in the functional space [see Eqs. (7) and (10)]. The last transformation involves only A , and it is known how to go over from integration of a gauge invariant function over A to the integration of the same function over the Wilson loop w (see, for example, Ref. [7]). Intuitively, to go over from the integration over A to the integration over w , we should replace the set of variables A with the parameters of the gauge transformations α , and with the gauge invariant w . It may seem that α contains the same number of parameters as A does, and we do not need w . That would be the case if there would be no gauge transformations that leave A invariant. In fact, the non-trivial transformations of A can be singled out by the requirement $\langle \chi_0 | \alpha \rangle = 0$, because the gauge transformation of A generated by χ_0 is trivial ($D\chi_0 = 0$). So, there is a correspondence between the variables encoded in A , and the variables defining a vector of the space of the gauge transformations transverse to χ_0 . The correspondence is given by the relation $\delta_\alpha A = D\alpha$. Therefore, the transformation of the variables from A to the variables parametrizing the gauge group generates the Jacobian $|\det[D']|$, where the prime denotes that the zero eigenvalue of D should be excluded from the computation of the determinant. It is known how to compute this determinant, see Ref. [7]. For $SU(2)$, $|\det[D']| = 1 - w^2$. The gauge invariant content of A is encoded in w , and the Jacobian of the transformation from the “gauge invariant piece of A ” to w can be computed in any convenient gauge, in particular, in the gauge where the eigenfunctions χ_p are given by Eq. (13). This extra Jacobian is $1/\sqrt{1-w^2}$. Therefore, the correct replacement of the measure of integration over A is as follows:

$$\prod_{x,y} dA(x,y) \rightarrow \prod_y \sqrt{1-w^2(y)} dw(y). \quad (16)$$

We stress that this substitution is valid in the integrals of gauge invariant functions. Another point to make is that the integration over each $w(y)$ runs from -1 to 1 .

We summarize the above by stating that the partition function of the gluodynamics over $S^1 \times M$ is

$$Z = \int \prod_{p>0} \prod_{y \in M, i} d[\Psi_i^p(y)]^\dagger d\Psi_i^p(y) \prod_{y \in M, i} dG_i(y) \times \prod_{y \in M} dw(y) \sqrt{1-w^2(y)} \exp(iS_{\text{glue}}). \quad (17)$$

Here we used the fact that $\Psi_i^{-p} = [\Psi_i^p(y)]^\dagger$ to restrict the product over p to the positive values [recall that $p = (2\pi n + \sigma\phi)/L$].

To complete the picture, we have to express S_{glue} in terms of the new variables. As S_{glue} is expressible in terms of the field strength tensor $F_{\mu\nu}$, it suffices to express $F_{\mu\nu}^p = \langle \chi_p | F_{\mu\nu} \rangle$ in terms of the new variables. This will be done in the next section.

III. FIELD STRENGTHS IN TERMS OF THE NEW VARIABLES

To express the field strengths in terms of the gauge invariant Fourier modes, we first recall how A_i is related to Ψ_i^p :

$$A_i = \Psi_i + C_i, \quad (18)$$

where $\Psi_i \equiv \sum_{p \neq 0} \chi_p \Psi_i^p$, and C_i is defined in Eq. (8). Expressing $F_{\mu\nu}$ for the case when one of the indexes corresponds to the compactified direction is simpler than for F_{ij} . We start from this case. Consider $E_i \equiv F_{xi}$, where the index x corresponds to the compactified direction. In terms of the covariant derivative D and the gauge potential, E_i reads

$$E_i = DA_i - \partial_i A. \quad (19)$$

Substitute into this formulas (18) and (8) to obtain

$$E_i = D\Psi_i + DB_i - \partial_i A. \quad (20)$$

This is the case because the difference between C_i and B_i is proportional to χ_0 and, therefore, it is nullified by the action of D .

Now consider the projection of E_i onto χ_p :

$$E_i^p = \frac{p}{i} \Psi_i^p + \frac{p}{i} B_i^p - (\partial_i A)^p. \quad (21)$$

Check the definition (3) and observe that the second term in the RHS above cancels against the third term if $p \neq 0$. Therefore, $E_i^p = p\Psi_i^p/i$ if $p \neq 0$. If $p = 0$, the first two terms do not contribute, and, therefore, $E_i^0 = -(\partial_i A)^0$. This is a gauge invariant:

$$\begin{aligned} \delta_\alpha (\partial_i A)^0 &= \langle -ig[\chi_0, \alpha] | \partial_i A \rangle + \langle \chi_0 | -ig[\partial_i A, \alpha] \rangle \\ &\quad + \langle \chi_0 | D\partial_i \alpha \rangle \\ &= 0. \end{aligned}$$

The last equality holds because in the last term D nullifies χ_0 , and the first two terms are canceled against each other.

To compute $(\partial_i A)^0$ go over to the gauge where the eigenfunctions are given by Eq. (13) and obtain $(\partial_i A)^0 = -2\partial_i w / [g\sqrt{L(1-w^2)}]$. Ultimately, for the Fourier modes E_i^p we have

$$E_i^0 = \frac{2\partial_i w}{g\sqrt{L(1-w^2)}}, \quad (22)$$

$$E_i^{p \neq 0} = \frac{p}{i} \Psi_i^p. \quad (23)$$

The next step is to treat F_{ij} . Use (18) and expand F_{ij} in powers of Ψ_i :

$$F_{ij} = (\partial_i C_j - \partial_j C_i - ig[C_i, C_j]) + (D_i[C]\Psi_j - D_j[C]\Psi_i) - (ig[\Psi_i, \Psi_j]). \quad (24)$$

Since Ψ and F_{ij} transform uniformly under gauge transformations, each bracket in the RHS above transforms uniformly, and the Fourier modes of each set of brackets are separately gauge invariant. Therefore, to compute them, we can use any convenient gauge.

Start with the first brackets. In the gauge where the eigenfunctions are given by Eq. (13), $C_i = G_i \chi_0$, and χ_0 is independent of y^i . Therefore, in this gauge, the first bracket becomes $[\chi_0(\partial_i G_j - \partial_j G_i)]$. We see that only the zero mode of the first bracket is nonzero, and it equals $\partial_i G_j - \partial_j G_i$. It can be interpreted as the field strength of the Abelian gauge field G_i over the space M .

Consider the second bracket of the RHS of Eq. (24). Do the same trick: go over to the gauge where the eigenfunctions are given by Eq. (13) and compute Fourier modes $[D_i(C)\Psi_j]^p$ (for this bracket, both terms of the bracket have gauge invariant Fourier modes). In the calculation, use Eq. (12). The answer is

$$[D_i(C)\Psi_j]^p = \partial_i \Psi_j^p - i \frac{g \sigma_p}{\sqrt{L}} G_i \Psi_j^p,$$

which looks similar to covariant derivative $\mathcal{D}_i[G]$ of an Abelian theory acting on a vector matter field Ψ_j^p defined over M . In this, G_i is the Abelian gauge field, and the Abelian charge of the field Ψ_j^p is $g \sigma_p / \sqrt{L}$. We note here that the Abelian charge of fields Ψ_i^p vanishes for the fields with $\sigma_p = 0$ (these fields are the Fourier modes corresponding to the middle levels of the triples of the spectrum of iD).

Consider now the last brackets of the RHS of Eq. (24). Expand the Ψ fields involved in this bracket into Fourier modes, and use the above convenient gauge to compute the emerging commutators. The answer for them generalizes Eq. (12):

$$[\chi_{p_1}, \chi_{p_2}] = (-1)^{|\sigma_{p_1} - \sigma_{p_2}|} \frac{s(\sigma_{p_1} - \sigma_{p_2})}{\sqrt{L}} \chi_{p_1 + p_2}. \quad (25)$$

Here, $s(\sigma_{p_1} - \sigma_{p_2})$ is the sign of the difference of the signs if the latter is nonzero, and it is zero if the difference of the signs is zero. To simplify the formulas, we introduce a notation for the product $(-1)^{|\sigma_1 - \sigma_2|} s(\sigma_1 - \sigma_2)$ appearing in the RHS above:

$$(-1)^{|\sigma_1 - \sigma_2|} s(\sigma_1 - \sigma_2) \equiv f(\sigma_1 - \sigma_2). \quad (26)$$

With this expression, the Fourier modes of the last bracket of the RHS of Eq. (24) take the following form:

$$(ig[\Psi_i, \Psi_j])^p = ig \sum_{p_1} \frac{f(\sigma_{p_1} - \sigma_{p-p_1})}{\sqrt{L}} \Psi_i^{p_1} \Psi_j^{p-p_1}. \quad (27)$$

Gathering together the above information about the separate brackets of the RHS of Eq. (24), we obtain the ultimate expressions for the Fourier modes of F_{ij} :

$$F_{ij}^0 = \mathcal{F}_{ij} - ig \sum_p \frac{f(\sigma_p - \sigma_{-p})}{\sqrt{L}} \Psi_i^p \Psi_j^{-p}, \quad (28)$$

$$F_{ij}^{p \neq 0} = \mathcal{D}_i \Psi_j^p - \mathcal{D}_j \Psi_i^p - ig \sum_{p_1} \frac{f(\sigma_{p_1} - \sigma_{p-p_1})}{\sqrt{L}} \Psi_i^{p_1} \Psi_j^{p-p_1}. \quad (29)$$

Here, the Abelian strength tensor \mathcal{F}_{ij} and the Abelian covariant derivatives $\mathcal{D}_i \Psi_j^p$ are defined as follows:

$$\mathcal{F}_{ij} \equiv \partial_i G_j - \partial_j G_i, \quad (30)$$

$$\mathcal{D}_i \Psi_j^p \equiv \partial_i \Psi_j^p - ig^p G_i \Psi_j^p. \quad (31)$$

Above, the Abelian charge g^p is

$$g^p \equiv \frac{g \sigma_p}{\sqrt{L}}. \quad (32)$$

A comment is in order: we succeeded in expressing the $SU(2)$ gluodynamics in terms of (almost) gauge invariant variables. The original gauge fields are not involved in this formulation. As there are no gauge noninvariant quantities left in the formulation, a legitimate question then appears: where does the non-Abelian character of the theory reside in this formulation? The answer is that it resides in the particular field content of the formulation obtained. A similar situation appears when a gauge theory is restricted to a particular gauge: There is no explicit gauge invariance after a gauge has been fixed, yet the emerging formulation still represents completely the original non-Abelian gauge theory.

This completes the description of the Fourier components of the field strength tensor. Let us list the general properties of $SU(2)$ gluodynamics in the new variables.

It is an Abelian gauge field theory over M .

Apart from the Abelian gauge field, the model contains an infinite tower of vector matter fields, and a single neutral scalar field (the Wilson loop) taking values in the interval $(-1, 1)$.

Generally, there are cubic and quartic interactions between the matter fields.

There are charged and uncharged vector matter fields.

In the next section, we discuss the prospects of using the new field variables. In particular, we consider the case when the compactified direction is lightlike, and the overall space $S^1 \times M$ carries a Minkowski metric. We do this case because there is a hope based on Ref. [3] that not only the infinite tower of the matter fields present in the model, but even the number of excitations of separate fields can be consistently

bounded from above by specifying a value of the total momentum of the system in the compactified lightlike direction.

IV. DISCUSSION

The action and the partition function of gluodynamics in terms of the new variables depend on the relation between the metric on the initial space $S^1 \times M$ and the compactified direction. Two particularly interesting cases are when the metric is Euclidean, and when the metric is Minkowskian and the compactified direction is lightlike. The former is interesting because it treats the case of finite temperature (inverse radius of the circle). The latter is interesting because there is a hope based on Ref. [3] that restricting the value of the total momentum of the system along the compactified direction is a condition stringent enough to leave only a finite number of degrees of freedom. In that case, we would have a unique possibility to study the ultraviolet divergences of gluodynamics nonperturbatively in a framework of quantum mechanics of a finite number of degrees of freedom. Here we present only initial steps on realizing the second possibility with the Minkowskian metric.

Let the initial space $S^1 \times M$ be equipped with Minkowski metric $g_{\mu\nu}$ with a single timelike direction. Let the direction along the circle be lightlike. Let $M = R \times T$, where R is a lightlike line, and T is the transverse space (for the four-dimensional space-time, T is a two-dimensional Euclidean space). In this section, the coordinate along the circle is denoted as x^- . Let x^+ be the coordinate on R , and y^I be the coordinates on T . Let the nonzero components of the metric in the coordinates x^-, x^+, y^I be defined by $g_{+-} = g_{-+} = 1$ and $g_{IJ} = -\delta_{IJ}$.

Our fields $\Psi_{+,I}^p$, $G_{+,I}$, and w depend on x^+ and y^I . The light-front formulation suggests that we consider x^+ as dynamical time, and the transverse space T as the space over which the initial configurations of the fields are set.

Let us write down S_{glue} using the above Minkowski metric and the new field variables. In terms of $E_+ \equiv F_{-+}$, $E_I \equiv F_{-I}$, F_{+I} , and F_{IJ} , S_{glue} reads

$$S_{\text{glue}} = \int dx^+ dy \left[\frac{1}{2} \langle E_+ | E_+ \rangle + \langle E_I | F_{+I} \rangle - \frac{1}{4} \langle F_{IJ} | F_{IJ} \rangle \right]. \quad (33)$$

The integral in x^+ runs over R , and the integral in y runs over T [recall, that the integral in x^- over the circle and the trace are hidden in the definition (1) of the scalar product].

Use the complete orthonormal base χ_p to replace the scalar products in Eq. (33) with the sums of the Fourier modes, separate the contribution of the zero modes, take into account that the negative modes equal the complex conjugation of the positive modes, and obtain

$$S_{\text{glue}} = \int dx^+ dy \left[\frac{1}{2} (E_+^0)^2 + E_I^0 F_{+I}^0 - \frac{1}{4} (F_{IJ}^0)^2 + \sum_{p>0} \left(\left| E_+^p \right|^2 + 2 \text{Re}(E_I^{-p} F_{+I}^p) - \frac{1}{2} \left| F_{IJ}^p \right|^2 \right) \right]. \quad (34)$$

The next stage is to use the formulas from the previous section to express this action in terms of the Fourier data. Here we want to discuss two general properties of the emerging theory.

If x^+ is considered as time, Ψ_+^p and G_+ are nondynamical, because there is no time derivatives of those fields in the action. Ψ_+^p can be integrated out, because the action is quadratic in it, and the relevant quadratic form is nondegenerate. The action is linear in G_+ , and variation over it generates a Gauss law which suffices to express the derivatives of w in terms of Ψ_i^p . In this way, we arrive at a theory whose field content is Ψ_i^p , G_i , and the zero mode of w (which is the integral of w over the transverse space T). This is the theory to which we apply the condition of a given positive value of the momentum component of the system along the x^- direction.

It was pointed out in Ref. [8], that, without an ultraviolet regularization, the Gauss law of the light-front formulation has no solutions if localized charges are present on the transverse plane. The variables introduced above help to clarify the situation. There are solutions to the Gauss law, but, in the presence of charges on the transverse plane, they involve discontinuities of the Wilson loop w .

Generally, there are two options in such a situation. One is to consider an ultraviolet regularization replacing the transverse space T with a lattice. Another option is to introduce in the continuum theory the locations of the singularities of w on the transverse plane as dynamical variables.

We point out that the transverse lattice was introduced in Ref. [9] (see Ref. [10] for recent references). Our formulation provides a new option for introducing the transverse lattice in a gauge invariant way. It is natural to consider the matter fields Ψ_i^p as defined on the sites of the transverse lattice, and replace the fields G_i by the phases defined on the links. Further work is needed to develop this light-front program.

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